IB Mathematics AA HL

Internal Assessment

Integrals of the Gaussian Function

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Contents

1 Introduction

While studying for my AP Calculus exam, I came across a remarkably interesting question. I was given the slope field for the differential equation $F'(x) = e^{-x^2}$ and was asked for $\lim_{x\to\infty} F(x)$ for the condition $F(0) = 0$ (see figure 1). A slope field or direction field essentially plots short line segments of slope $F'(x)$ at selected points throughout the domain. Since any solution curve $F(x)$ passing through a point (x, y) must be tangent to the line segment plotted there, it is a useful tool that helps to visualise the solutions to a differential equation (Hass et al.).

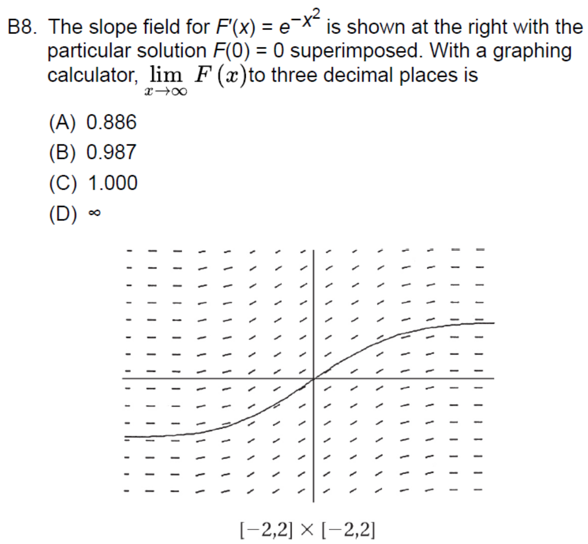

Figure 1. Question from Donovan et al.

While the question prompted me to use my graphing calculator, I had to try it out myself. All it looked like I had to do was compute the anti-derivative.

$$F(x) = \int e^{-x^2} dx. \tag{1.1}$$

A substitution was not leading to a solution, so I considered integration by parts.

Let $u = e^{-x^2}$ and $dv = dx$, so that:

$$du = -2xe^{-x^2}\, dx, \tag{1.2}$$

and

$$v = x, \tag{1.3}$$

giving,

$$x\ e^{-x^2} - \int e^{-x^2}\left(-2xe^{-x^2}\right) dx \tag{1.4}$$

This just seemed to make the integral harder. I started having doubts about the existence of a solution and looked online for an answer. I found a post on Stack Exchange talking about the integral of this very function - the ”Gaussian function” (Zolani13). I found out there is no solution to the indefinite integral that can be expressed in terms of ‘elementary’ functions. Upon further research, I discovered that this lack of an elementary solution to the indefinite integral of the Gaussian function is even mathematically provable by Liouville’s Theorem (Conrad). But clearly the graph on the slope field in figure 1 shows that a solution exists. So then, *how are these analytically unsolvable integrals solved?*

Interestingly however, it is possible to analytically compute the definite integral over the real line, $\int_{-\infty}^{\infty} e^{-x^2} dx$. In fact, it is equal to $\sqrt{\pi}$. This is known as the Gaussian Integral.

Theorem 1.1.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{1.5}$$

What a weird place to find π, isn’t it? But as mathematics Youtuber Grant Sanderson says, ”Whenever you see pi show up in Math, there’s always going to be a circle hiding around somewhere.” (Sanderson). In my Mathematics Investigation, I would like to *figure out this connection between the Gaussian Integral and π*, as well as *explore methods of evaluting analytically unsolvable integrals over finite intervals via the definite Gaussian integral.*

2 Gaussian Integral Over The Real Line

In this section, we will go over methods of evaluating the Gaussian Integral over the real line and explore its relationship to π.

2.1 Graphing the Gaussian Function

Graphing out the function $f(x) = e^{-x^2}$ (see Figure 2), we see that it has a bell curve-like shape, symmetric across the y-axis. The symmetry is because the x^2 term is the same regardless if the input is positive or negative. As a result, both $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. Hence the x-axis, $y = 0$ can be considered an asymptote. It peaks at $x = 0$ with output $f(0) = 1$. There are still no obvious circles, but the symmetry turns out to be important.

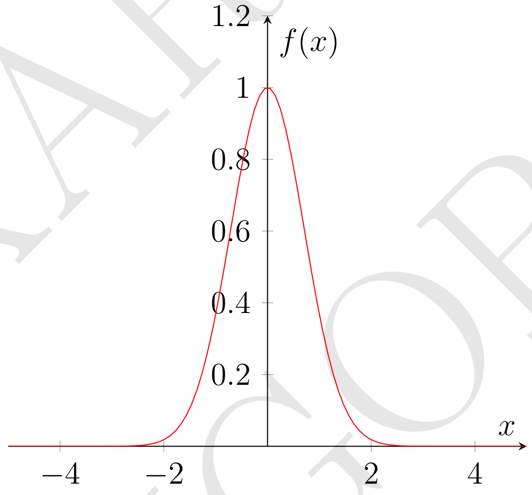

Figure 2. Graph of $f(x) = e^{-x^2}$ plotted with Ti*k*Z & PGF (version 1.18.1)

2.2 From Single-Variable to Multi-Variable Calculus

On the Stack Exchange post, a comment by user Faust suggested squaring the integral. Since the evaluated integral yields $\sqrt{\pi}$, the square of the integral yields π. This might make the connection to circles more evident.

Let

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx, \tag{2.1}$$

Then,

$$\begin{aligned} I^2 &= (\int_{-\infty}^{\infty} e^{-x^2} dx)^2 \\ &= (\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-x^2} dx). \end{aligned} \tag{2.2}$$

Since both integrals are definite, we can conclude that the choice of variable does not impact the evaluation of the integral. Hence, we can convert this single variable calculus problem into a multi-variable calculus problem by making one of the integrals in the product with respect to y instead of x. Multi-variable calculus, as its name suggests, is the application of calculus to functions of more than one variable. In this case, the variables x and y.

$$I^2 = (\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-y^2} dy). \tag{2.3}$$

This can then be rewritten as the double integral

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy \tag{2.4}$$

While this may appear like an abuse of notation. It can be proven that the two are equivalent.

Lemma 2.1.

$$(\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-y^2} dy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy. \tag{2.5}$$

Proof.

$$\begin{aligned} RHS &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy \\ &= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx) dy \end{aligned} \tag{2.6}$$

Noticing that the e^{-y^2} term is independent of x,

$$\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx) dy = \int_{-\infty}^{\infty} e^{-y^2} (\int_{-\infty}^{\infty} e^{-x^2} dx) dy \tag{2.7}$$

Noticing again that the $\int_{-\infty}^{\infty} e^{-x^2} dx$ term is independent of y,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} (\int_{-\infty}^{\infty} e^{-x^2} dx) dy &= (\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-y^2} dy) \\ &= LHS \end{aligned} \tag{2.8}$$

□

The double integral can be further simplified to give

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy \tag{2.9}$$

The function being integrated in equation 2.9 is the 2-Dimensional Gaussian Function.

2.3 Analysing the 2-Dimensional Gaussian Function

In the 2-Dimensional Gaussian Function, we can observe the expression $x^2 + y^2$, which also shows up in the equation of a circle. For each constant r^2, the values of x and y that satisfy $x^2 + y^2 = r^2$ form a circle with radius r (see figure 3).

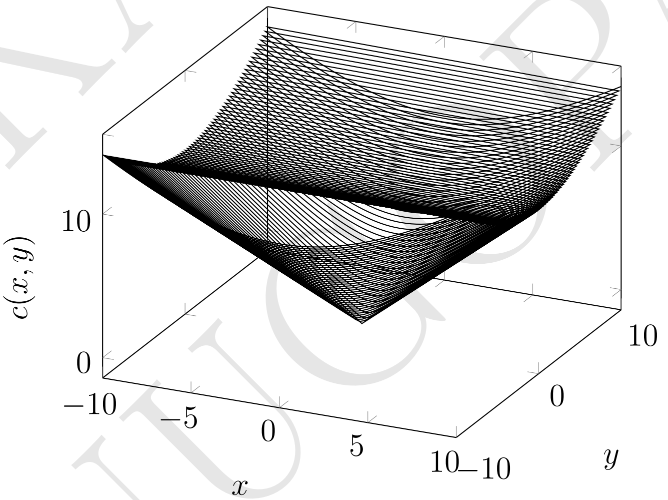

Figure 3. Graph of $c(x, y) = \sqrt{x^2 + y^2}$ plotted with Ti*k*Z & PGF (version 1.18.1)

Hence, the points which are same radius away from the origin in the 2-Dimensional Gaussian Function have the same value of $x^2 + y^2$. As there is no other dependence on x or y in the function, it is rotationally symmetric about the z-axis (see figure 4). Note that the double integral in Equation 2.9 corresponds to the volume under the surface in the figure below.

These facts are of key importance, as they means that the intersection of the 2-Dimensional Gaussian function with any slice in the z-axis is a circle. Hence, we can proceed to solve the integral by simply finding the volume of revolution of the 1-Dimensional Gaussian function with domain $[0, \infty)$ about the z-axis.

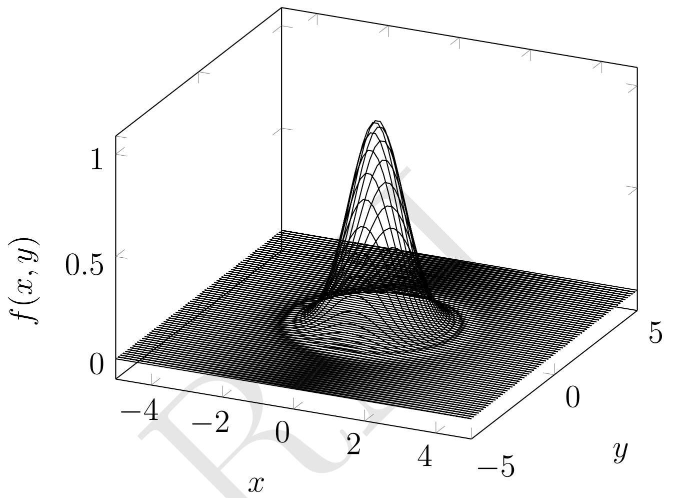

Figure 4. Graph of $f(x, y) = e^{-(x^2+y^2)}$ plotted with Ti*k*Z & PGF (version 1.18.1)

2.4 Evaluating by Volume of Revolution

First, we have to find the inverse function, $f^{-1}(x)$, for the 1-Dimension Gaussian Function with the domain $[0, \infty)$.

Let

$$y = e^{-x^2}. \tag{2.10}$$

Taking ln on both sides,

$$\begin{aligned} \ln y &= \ln e^{-x^2} \\ \ln y &= -x^2. \end{aligned} \tag{2.11}$$

Making x the subject of the equation,

$$\begin{aligned} -\ln y &= x^2 \\ \sqrt{-\ln y} &= x. \end{aligned} \tag{2.12}$$

Hence $f^{-1}(x) = \sqrt{-\ln x}$.

To obtain the volume of the solid of revolution, the following integral must be evaluated:

$$\int_0^1 \pi x^2 dx \tag{2.13}$$

Substituting x and then solving,

$$\begin{aligned}\int_0^1 \pi(\sqrt{-\ln x})^2 dx &= \int_0^1 -\pi \ln x dx \\ &= -\pi \int_0^1 \ln x dx \\ &= -\pi \Big| x \ln x - x \Big|_0^1 \\ &= -\pi[1 \times \ln 1 - 1 - (0 \times \ln 0 - 0)] \\ &= -\pi[1 \times 0 - 1 - 0] \\ &= -\pi[-1] \\ &= \pi.\end{aligned} \tag{2.14}$$

Now,

$$\begin{aligned}I^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^1 \pi(\sqrt{-\ln x})^2 dx \\ &= \pi.\end{aligned} \tag{2.15}$$

It follows that,

$$I = \sqrt{\pi}, \tag{2.16}$$

proving Theorem 1.1.

2.5 Evaluating with Polar Coordinates

The rotational symmetry in the 2-Dimensional Gaussian function also makes it a good candidiate to be expressed in terms of polar coordinates.

In the 2-Dimension Cartesian coordinate system, every point is described by a pair (x, y), where x is the horizontal distance from the origin and y is the vertical distance from the origin. While we are often used to thinking about functions in a way where every x-coordinate input is associated with a y-coordinate output, it is also possible for points to be described in other ways.

In the polar coordinate system, every point is described by the pair (r, θ), where r is the distance from the origin and θ is the angle from the positive x-axis. The polar coordinate system is to the Cartesian coordinates system what the mod-arg form of complex numbers is to the rectangular form. However here, instead of using the imaginary part as an axis, we use another variable. Hence, the conversion from Cartesian coordinates to Polar coordinates is similar to the conversion from the Rectangular form to the mod-arg form.

$$x = r\cos\theta. \tag{2.17}$$

$$y = r\sin\theta. \tag{2.18}$$

Hence,

$$\begin{aligned} x^2 + y^2 &= r^2\sin^2\theta + r^2\cos^2\theta \\ &= r^2(\sin^2\theta + \cos^2\theta). \end{aligned} \tag{2.19}$$

Applying the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$,

$$x^2 + y^2 = r^2. \tag{2.20}$$

Combining and rearranging for θ,

$$\theta = \arctan\frac{y}{x}. \tag{2.21}$$

Do note that the 2-Dimensional Gaussian function, $f(x, y)$ depends on both x and y. Hence the output is usually plotted in the 3rd dimension, z, the distance perpendicular to the $xy-$plane or height. Converting the input to polar coordinates changes the form of the input, but the output is still the height in the z-axis. This is often called the cylindrical coordinate system where every point in a 3-Dimensional space can be described by the coordinates (r, θ, z) instead of the Cartesian coordinates (x, y, z).

Geometrically, the term $dxdy$ in the double integral from Equation 2.9 can be interpreted as an infinitesimal rectangle of area dA, where dA is the infinitesimal change in the area, corresponding to infinitesimal changes dx and dy.

To convert this area element $dxdy$ into the polar coordinates, let us consider Figure

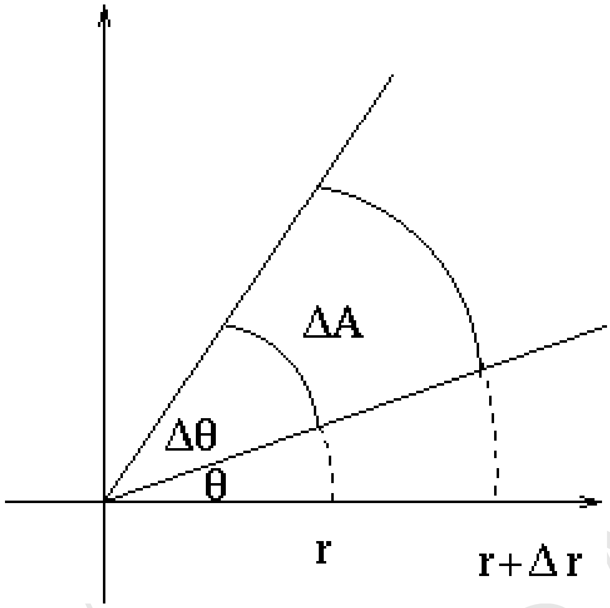

Figure 5. From "The area element in polar coordinates"

5. For the small changes $\Delta\theta$ and Δr, the corresponding change in the area is ΔA. In the limit, as $\Delta\theta$ and Δr get infinitesimally small, ΔA can be approximated as the area of the rectangle with sides Δr and, by the arc length formula, $r\Delta\theta$. Hence,

$$dxdy = dA = rd\theta dr. \tag{2.22}$$

Substituting Equations 2.21 and 2.20 into the double integral from Equation 2.9 and changing the limits appropriately,

$$I^2 = \int_0^\infty \int_0^{2\pi} e^{-r^2} rd\theta dr. \tag{2.23}$$

Since we are integrating over the entire surface, the limits of the integral are the entire polar domain. For one full rotation, $0 \leq \theta \leq 2\pi$, and for the radius, $r \geq 0$.

Noticing that $e^{-r^2}r$ is independent of $d\theta$,

$$I^2 = \int_0^\infty e^{-r^2} r \int_0^{2\pi} d\theta dr. \tag{2.24}$$

Evaluating the inner integral with respect to $d\theta$,

$$\begin{aligned} I^2 &= \int_0^\infty e^{-r^2} r \Big|\theta\Big|_0^{2\pi} dr \\ &= \int_0^\infty e^{-r^2} r[2\pi - 0] dr \\ &= 2\pi \int_0^\infty e^{-r^2} r dr \end{aligned} \tag{2.25}$$

To evaluate the outer integral with respect to dr a subsitution can be used.

Letting $u = -r^2$,

$$\frac{du}{dr} = -2r \tag{2.26}$$

It follows that,

$$-\frac{1}{2} du = r dr. \tag{2.27}$$

The new limits of the integral would be,

$$\begin{aligned} u(0) &= 0. \\ u(\infty) &= -\infty. \end{aligned} \tag{2.28}$$

Carrying out the substitution and integrating,

$$\begin{aligned} I^2 &= 2\pi \int_0^{-\infty} e^u (-\frac{1}{2} du) \\ &= -\pi \int_0^{-\infty} e^u du \\ &= -\pi \int_0^{-\infty} e^u du \\ &= -\pi \Big| e^u \Big|_0^{-\infty}. \end{aligned} \tag{2.29}$$

Substituting u back into Equation 2.28,

$$I^2 = -\pi \Big| e^{-x^2} \Big|_0^{-\infty}. \tag{2.30}$$

Since the integral is improper,

$$\begin{aligned} I^2 &= \lim_{t \to -\infty} [-\pi \Big| e^{-x^2} \Big|_0^t] \\ &= \lim_{t \to -\infty} [-\pi[e^{-t^2} - e^{-(0)^2}]] \\ &= -\pi[\lim_{t \to -\infty} e^{-t^2} - 1] \\ &= -\pi[0 - 1] \\ &= \pi. \end{aligned} \tag{2.31}$$

Finally,

$$I = \sqrt{\pi}. \tag{2.32}$$

I find it intriguing how converting this seemingly impossible single-variable calculus problem into a multivariable calculus problem is a rather counter-intuitive step. At first, it feels like it is making the problem more complicated, but eventually it opens not just one, but many avenues to solve it. Perhaps this is one of the reasons why pursuing deeper knowledge of Mathematics is always important. Even if it might not be directly obvious how a new mathematical concept we learn could ever help me, it might always prove useful in solving problems that I previously thought unsolvable.

This particular trick of squaring the integral and then using multivariable calculus to solve it has been attributed to legendary mathematician Poisson (Bell). While de Moivre first discovered the integral in 1721, this elegant method of solving the integral was only published by Poisson in 1835 (Stahl; Denis)!

3 Gaussian Integrals Over Finite Limits

In the previous section, we explored the definite integral of the Gaussian function over the interval $(-\infty, \infty)$. However, what about the definite integral over an arbitrary finite interval $[a, b]$? Since there is no elementary anti-derivative for this integral, the definite integral with finite limits can only be approximated.

For this section, we will try to evaluate the following definite integral using various method of approximation.

$$\int_1^2 e^{-x^2} dx. \tag{3.1}$$

3.1 Approximation by Taylor Series

Having learnt about the MacLaurin series, it was the first thing that came to my mind when I was thinking of methods to approximate the integral. Since polynomials are fairly easy to integrate, expressing the Gaussian function as a polynomial would be helpful in finding an approximation. However, the MacLaurin series only approximates a function around $x = 0$. A more general form of the MacLaurin series is the Taylor series.

Definition 3.1 (Taylor Series)**.** For a function $f(x)$ that is infinitely differentiable at $x = a$, the Taylor series centered at a is given by (Libretexts, "8.8: Taylor series"),

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \tag{3.2}$$

The Taylor series is more suitable than the Maclaurin series for approximations of definite integrals over any arbitrary finite interval as it allows us to choose the point around which the function is being approximated. For an intuition for how the Taylor series does this, we can notice that the constant a is essentially shifting the MacLaurin approximation to the left by a.

Since we are attempting to evaluate the integral over the interval $[1, 2]$, it would make sense to make this polynomial approximation somewhere near it. Applying the Taylor series expansion to the Gaussian function centred at $x = 1$,

$$\begin{aligned} e^{-x^2} &= e^{-1} + \frac{d(e^{-x^2})}{dx}\Big|_{x=1}(x-1) + \frac{1}{2}\frac{d^2(e^{-x^2})}{dx^2}\Big|_{x=1}(x-1)^2 + \\ &\quad \frac{1}{6}\frac{d^3(e^{-x^2})}{dx^3}\Big|_{x=1}(x-1)^3 + \frac{1}{24}\frac{d^4(e^{-x^2})}{dx^4}\Big|_{x=1}(x-1)^4 + \\ &\quad \frac{1}{120}\frac{d^5(e^{-x^2})}{dx^5}\Big|_{x=1}(x-1)^5 + \cdots \\ &= e^{-1} - \frac{2}{e}(x-1) + \frac{1}{e}(x-1)^2 + \frac{2}{3e}(x-1)^3 - \frac{5}{6e}(x-1)^4 + \frac{1}{15e}(x-1)^5 + \cdots . \end{aligned} \tag{3.3}$$

Truncating to 5 terms as an approximation:

$$e^{-x^2} \approx e^{-1} - \frac{2}{e}(x-1) + \frac{1}{e}(x-1)^2 + \frac{2}{3e}(x-1)^3 - \frac{5}{6e}(x-1)^4. \tag{3.4}$$

The integral in Expression 3.1 can now be approximated by integrating Equation 3.4.

$$\int_1^2 e^{-x^2} \approx \int_1^2 [e^{-1} - \frac{2}{e}(x-1) + \frac{1}{e}(x-1)^2 + \frac{2}{3e}(x-1)^3 - \frac{5}{6e}(x-1)^4]dx. \tag{3.5}$$

By linearity,

$$\begin{aligned} \int_1^2 e^{-x^2} &\approx \int_1^2 e^{-1}dx - \int_1^2 \frac{2}{e}(x-1)dx + \int_1^2 \frac{1}{e}(x-1)^2dx + \\ &\qquad \int_1^2 \frac{2}{3e}(x-1)^3dx - \int_1^2 \frac{5}{6e}(x-1)^4dx \\ &\approx e^{-1}\int_1^2 1dx - \frac{2}{e}\int_1^2 (x-1)dx + \frac{1}{e}\int_1^2 (x-1)^2dx + \\ &\qquad \frac{2}{3e}\int_1^2 (x-1)^3dx - \frac{5}{6e}\int_1^2 (x-1)^4dx \end{aligned} \tag{3.6}$$

Integrating,

$$\begin{aligned} &e^{-1}\int_1^2 1dx - \frac{2}{e}\int_1^2 (x-1)dx + \frac{1}{e}\int_1^2 (x-1)^2dx + \frac{2}{3e}\int_1^2 (x-1)^3dx + \\ &\qquad \frac{2}{3e}\int_1^2 (x-1)^3dx - \frac{5}{6e}\int_1^2 (x-1)^4dx \\ &= e^{-1}\Big[x\Big]_1^2 - \frac{2}{e}\Big[\frac{(x-1)^2}{2}\Big]_1^2 + \frac{1}{e}\Big[\frac{(x-1)^3}{3}\Big]_1^2 + \frac{2}{3e}\Big[\frac{(x-1)^4}{4}\Big]_1^2 - \frac{5}{6e}\Big[\frac{(x-1)^4}{4}\Big]_1^2. \end{aligned} \tag{3.7}$$

Evaluating,

$$\begin{aligned} &\frac{1}{e}\Big[2-1\Big] - \frac{2}{e}\Big[\frac{(1)^2}{2} - \frac{(0)^2}{2}\Big] + \frac{1}{e}\Big[\frac{(1)^3}{3} - \frac{(0)^3}{3}\Big] + \frac{2}{3e}\Big[\frac{(1)^4}{4} - \frac{(0)^4}{4}\Big] - \frac{5}{6e}\Big[\frac{(1)^5}{5} - \frac{(0)^5}{5}\Big] \\ &= \frac{1}{e} - \frac{2}{e}\Big[\frac{1}{2}\Big] + \frac{1}{e}\Big[\frac{1}{3}\Big] + \frac{2}{3e}\Big[\frac{1}{4}\Big] - \frac{5}{6e}\Big[\frac{1}{5}\Big] \\ &\approx 0.122626. \end{aligned} \tag{3.8}$$

This value is fairly close to the value given by a calculator, 0.13526. However, one must always remember that calculators too often only use approximations. To determine the error of this approximation, we can use a more rigorous method - the Lagrange Error Bound. This can be derived from Taylor's Theorem.

Theorem 3.2 (Taylor's Theorem)**.** *If a function f and its first $(n+1)$ derivatives are continuous on an interval between a and x then for each x in this interval*

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \\ &= T_n + R_n. \end{aligned} \tag{3.9}$$

where

$$R_n = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!}, \tag{3.10}$$

T_n is the Taylor expansion with n terms and c is some number between a and x.

R_n is known as the Lagrange remainder. Taylor's theorem essentially expresses a function as the sum of a truncated Taylor series expansion and the corresponding error for that polynomial (Donovan et al.).

Theorem 3.3 (Lagrange Error Bound)**.** *It is not possible to determine R_n without knowing the exact value of c. Hence, it is easier instead to find an upper bound for the value of R_n by choosing a value of c between a and x such that R_n is maximised, such that:*

$$R_n \leq \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!}. \tag{3.11}$$

This upper bound is known as the Lagrange Error Bound.

By Taylor's Theorem, for the approximation in Equation 3.8,

$$\int_1^2 e^{-x^2} dx \approx \int_1^2 T_5 dx + \int_1^2 R_5 dx. \tag{3.12}$$

Applying the Lagrange Error Bound on R_5,

$$\int_1^2 R_5 dx \leq \int_1^2 \frac{f^5(c)(x-1)^5}{5!} dx. \tag{3.13}$$

Finding $f^5(c)$,

$$\int_1^2 R_5 dx \leq \int_1^2 \frac{-(32c^5 - 160c^3 + 120c)e^{-c^2}}{120}(x-1)^5 dx. \tag{3.14}$$

The value of c between 1 and 2 for which $f^5(c)$ is maximised is $c = 1.336$. It follows,

$$\int_1^2 R_5 dx \leq \frac{14.267}{120}\Big[\frac{(x-1)^6}{6}\Big]_1^2. \tag{3.15}$$

Evaluating,

$$\int_1^2 R_5 dx \leq 0.019815. \tag{3.16}$$

Therefore, by the Lagrange Error Bound, the approximation $\int_1^2 e^{-x^2} dx \approx 0.122626$ is accurate to an error of 0.019815. It is quite interesting how we can mathematically determine the error of this approximation. However, since it only an upper bound, it does lead one to ask: what is the error of the error? That is to say, how close is this error to the actual error? Perhaps this is not a question that can be answered all the time. Even mathematics has uncertainty sometimes.

The choice to take the Taylor series expansion around $x = 1$, instead of $x = 2$ or some other value between 1 and 2 was somewhat arbitrary. It would be an intriguing extension to consider how the error varies based on the point where we choose to take the Taylor series expansion, even outside the interval of this integral. This can be the subject of a future investigation.

3.2 Approximation by Euler Method

Another method of approximating definite integrals is by the Euler method. The Euler method is a numerical method that can be used to solve Ordinary Differential Equations. It works by repeated approximating subsequent values of the solution based on an initial condition and the differential equation itself.

While it is primarily used to solve Ordinary Differential Equations, we can always use the First Fundamental Theorem of Calculus to reframe an integral as a differential equation and the subsequently use the Second Fundamental Theorem of Calculus to evaluate Expression 3.1.

Theorem 3.4 (First Fundamental Theorem of Calculus)**.** *If a function $f(x)$ is continuous on an open interval $[a, b]$ and a function $F(x)$ is defined by*

$$F(x) = \int_a^x f(t)dt, \tag{3.17}$$

then

$$F'(x) = f(x), \tag{3.18}$$

over interval $[a, b]$

Theorem 3.5 (Second Fundamental Theorem of Calculus)**.** *For a function $f(x)$ that is continuous over the interval $[a, b]$, where $F(x)$ is an antiderivative of $f(x)$ (Libretexts, "5.3: The fundamental theorem of calculus"),*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{3.19}$$

In this case, by the First Fundamental Theorem of Calculus, if $F(x)$ is defined by,

$$F(x) = \int_0^x e^{-t^2} dt, \tag{3.20}$$

then

$$F'(x) = e^{-x^2}. \tag{3.21}$$

This brings us back to the original differential equation provided in the question in the Introduction section. Let us continue to consider the condition $F(0) = 0$. Rewriting the condition as the initial value

$$y_0 = 0, \tag{3.22}$$

we can now use the Euler method to numerically find subsequent values of y.

Recall that

$$y_{n+1} = y_n + hF'(x_n), \tag{3.23}$$

where h is the step size (Awada et al.).

For step size $h = 0.25$,

$$y_1 = 0 + (0.25)(e^{-0^2}) \quad = 1. \tag{3.24}$$

Likewise,

$$y_2 = 0.25 + (0.25)(e^{-0.25^2}) \quad = 0.484853. \tag{3.25}$$

Continuing the algorithm until $n = 8$,

n	x	y_n	$F'(x)$
0	0	0	1
1	0.25	0.25	0.939413
2	0.5	0.484853	0.778801
3	0.75	0.679553	0.569783
4	1	0.821999	0.367879
5	1.25	0.913969	0.209611
6	1.5	0.966372	0.105399
7	1.75	0.992722	0.046771
8	2	1.004414	0.018316

Table 1. Using the Euler method to Approximate Values of $F(x)$

By the Second Fundamental Theorem of Calculus, evaluating $F(2) - F(1)$ yields the area $y_8 - y_1 = 0.182415$. Since step size h chosen is relatively large, this value is more inaccurate compared to the value calculated in the previous subsection. Nonetheless, the Euler method is a simple way to compute definite integrals without analytical solutions. Its recursive, algorithmic nature makes it very suitable for use with computers, something that we will exploit in the next section.

Euler method is just one of out a vast toolbox of mumerical methods. Many other numerical methods such as the Implicit Euler method and the Crank–Nicolson method are derived from the Euler method above (Marini). The vast body of research conducted on numerical methods is a testament to the prevalence of unsolvable differential equations in the world. From quantum mechanics to blood flow, they are involved in many real world problems (Alsemiry et al.). It could be a subject of future study - studying the impact of the numerical method used on the solution obtained. Choosing the right numerical method to use can be an important decision to make when solving real world problems, especially when small inaccuracies could lead to big issues.

3.3 Graphing the Anti-Derivative

One might notice that using the Euler method, we get the approximated values of y at equally spaced intervals. This makes it a way one can plot the antiderivative of a function even if it cannot be expressed in terms of elementary functions (see figure 6).

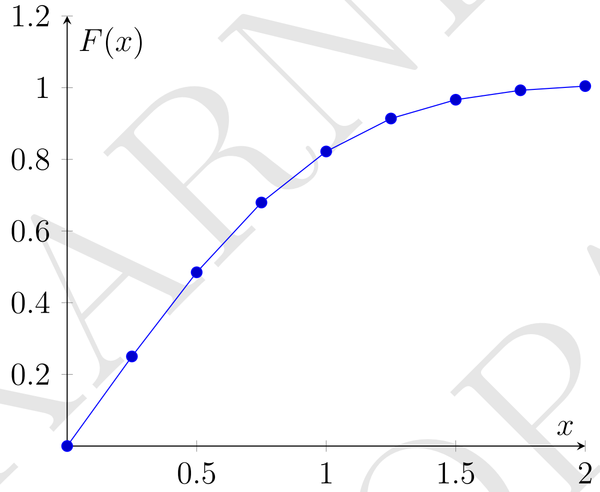

Figure 6. Approximation of $F(x) = \int_0^x e^{-t^2} dt$ for $h = 0.25$ over domain $[0, 2]$ plotted with Ti*k*Z & PGF (version 1.18.1)

The Euler method can also be used backwards to approximate function $F(x)$ for negative x. Microsoft Excel was used to do this.

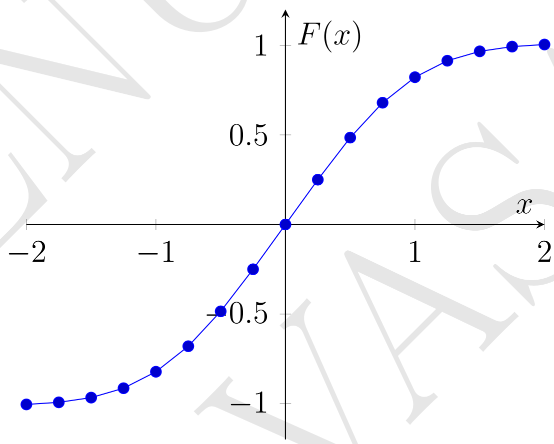

Figure 7. Approximation of $F(x) = \int_0^x e^{-t^2} dt$ for $h = 0.25$ over domain $[-2, 2]$ plotted with Ti*k*Z & PGF (version 1.18.1)

This was the function plotted in the question shown in the introduction.

3.4 The Error Function

Definition 3.6. The anti-derivative plotted above, when rescaled by a factor of $\frac{2}{\sqrt{\pi}}$, is known as the error function, $\mathrm{erf}(x)$ (Weisstein).

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{3.26}$$

While I have made several mentions of the phrase "elementary functions" in the investigation, what exactly constitutes the elementary functions? The error function is not considered an elementary function but it still plays a crucial role in mathematics. For example, in Probability Theory, the error function is encountered in the cumulative distribution function for a normal (also known as Gaussian) distribution (Weisstein). It also appears in solutions to Diffusion problems in Physics (Eagar and Allen).

This way, calculus opens new doors for mathematics. In this case, the integral of the Gaussian function leads to an important function that would otherwise have been impossible to express. I wonder if there are any other such functions whose definitions include integrals of elementary functions. I am also curious about other concepts in the field of mathematics that can create new dimensions to explore - new functions that we may have never seen before. What about functions which have domains that expand to complex numbers? There are certainly a lot of possibilities to build upon the fundamentals learnt in IB.

4 Conclusion

This function really is a rabbit hole with so much to uncover even beyond these pages. It is fascinating how Euler's number, Pi, calculus and infinity all come together in just one equation - the Gaussian Integral. Through both the rotational symmetry and polar coordinates, we were able to determine the connection between this integral and pi. The short detour into multivariable calculus was rather informative - sometimes seemingly simple problems need unexpected tools to crack. I realised it is not always easy to know which tools to use and how. There is an element of creativity involved in mathematics to think out of the box and reframe a question the way it is easiest to solve.

It was also interesting to come so close to the limitations of mathematics. Some functions simply cannot be integrated conventionally and some functions cannot be expressed very easily even if they are useful. Moreover, sometimes mathematicians might be limited by the mathematics that has been discovered so far. It was very enlightening to see how mathematicians work their way out of these situations, for example, by exploring rigorous methods of approximation or defining entirely new functions. After all, it is this work that paves the way for deeper discoveries in the future.

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