**The Relationship Between Oscillating Length, Period and the Young's Modulus of an Oscillating Cantilever**

# 1 Introduction

A cantilever is a beam that is supported at one end and carrying a load at the other end (Britannica). The oscillating cantilever is a great practical example of harmonic motion. It has uses on the wings on planes to cantilever bridges to diving boards to Nanotechnology (Bayraktar and Demirtaş; Lang et al.). I was particularly drawn to the use of cantilevers in overhanging structures. I always thought it was amazing how structures such as the Marina Bay Sands are able to support such a large amount of mass hanging seemingly mid-air - even with the forces from the wind and people on it! Clearly engineers must account for forces that might be applied to these cantilevers which may result in motion.

The oscillatory motion of a cantilever has many factors affecting it: the length of the cantilever, the mass it must support, the stiffness of the cantilever, the shape and size of the cantilever. In realistic situations, there are several other factors too: damping, air resistance, the uniformity of the mass distribution in the cantilever as well as others depending on the use case. For the purpose of simplification, in this investigation I will be considering a light cantilever oscillating with a mass concentrated on the free end, neglecting air resistance. These assumptions will be discussed in further detail in the Evaluation section. Some possible dependent variables are the amplitude of the deflection during oscillation and the period of oscillation.

The variables I chose to investigate are the oscillating length and period of oscillation. This led to the research question:

*What is the effect of oscillating length on the period of a light oscillating cantilever?*

# 2 Theoretical Model

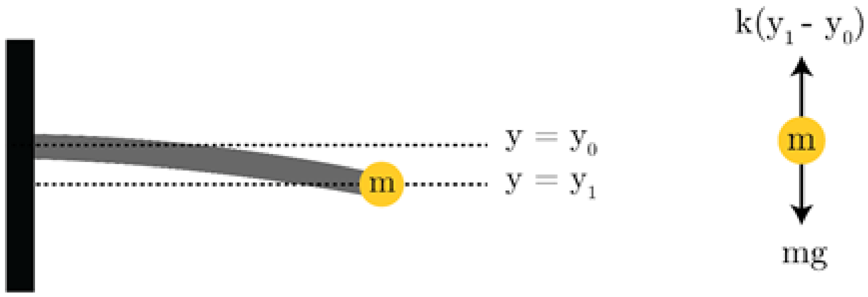
A light oscillating cantilever with a point mass $m$ at the end can be modelled with Hooke's Law for small oscillations (NT-MDT).

Consider the equilibrium position $y_1$ after a mass is attached to the free end of the cantilever as depicted in Figure 1a. By Newton's 2nd law we have

$$mg = k\left(y_1 - y_0\right), \tag{2.1}$$

where $k$ is a spring constant given by the stiffness of the cantilever (NT-MDT) with units

$\mathrm{N\,m^{-1}}$.




(a) Deflection From Initial Position

(b) Free Body Diagram

Figure 1. Modelling Cantilever With Hooke's Law. Images created with Adobe Illustrator.

At other displacements, $y$, the net force is given by

$$m\ddot{y} = mg - k\left(y - y_0\right). \tag{2.2}$$

Substituting the equilibrium equation:

$$m\ddot{y} = k\left(y_1 - y_0\right) - k\left(y - y_0\right) \tag{2.3}$$

$$m\ddot{y} = k(y_1 - y). \tag{2.4}$$

If the axis is defined such that the initial position $y_1 = 0$, then

$$m\ddot{y} = -ky \tag{2.5}$$

$$\ddot{y} = -\frac{k}{m}y. \tag{2.6}$$

In the case of the oscillating cantilever, the stiffness $k$ is given by

$$k = \frac{3EI}{l^3}, \tag{2.7}$$

where $E$ is the Young's Modulus of the cantilever with units $\mathrm{N\,m^{-2}}$, $I$ is the Area Moment of Inertia with units $\mathrm{m^4}$ and $l$ is the length of the cantilever in metres(Anand and Parks). For a cuboid cantilever (Elliott):

$$I = \frac{bh^3}{12}, \tag{2.8}$$

where $b$ is the width in metres and $h$ is the height in metres of a cross-section of the cantilever.

Substituting equation 2.8 into 2.7:

$$k = \frac{Ebh^3}{4l^3}. \tag{2.9}$$

Substituting equation 2.9 into 2.6:

$$\ddot{y} = -\frac{Ebh^3}{4ml^3}y. \tag{2.10}$$

Comparing equation 2.10 with the defining equation of Simple Harmonic Motion $\ddot{y} = -w^2y$, we obtain the angular frequency, $\omega$, of the system in $\mathrm{rad\,s^{-1}}$:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Ebh^3}{4ml^3}}, \tag{2.11}$$

where $T$ is the period of one complete oscillation in seconds.

Making $T$ the subject of the above equation, we have

$$T^2 = \frac{16\pi^2 m}{Ebh^3}l^3. \tag{2.12}$$

From here, it follows that

$$T^2 \propto l^3. \tag{2.13}$$

Interestingly, this relationship is very similar to Kepler's Third Law, $T^2 \propto r^3$, in which $r$ is the mean orbital radius of a planet and $T$ is the period of an orbit (Karttunen 119). From this relationship, I hypothesize that $T$ will be positively correlated to $l$ and that the graph of $T^2$ against $l^3$ will be linear, where the constant of proportionality is

$$\frac{16\pi^2 m}{Ebh^3}. \tag{2.14}$$

All the variables in this constant are directly measurable, with exception of Young's Modulus $E$. Hence, based on this result, the value of $E$ of a specific cantilever can hypothetically be experimentally determined from the gradient of a graph plotting $T^2$ against $l^3$. This can be used to further verify the validity of this model by comparing the theoretical value obtained with literature values.

# 3 Variables

**Independent Variable:** Oscillating Length (m)

A 0.600m steel ruler is used for the cantilever. The freely suspended oscillating length can be measuring directly from the readings on the ruler. The range used will be between 0.300m to 0.600m with intervals of 0.050m. The lower bound is to allow for a consistent initial displacement – for lengths less than 0.300m, the cantilever is not flexible enough for the free end to be displaced sufficiently.

**Dependent Variable:** Period of One Oscillation (s)

Since counting and timing the oscillations manually in real time is difficult for faster oscillations, it will be first recorded using a 30 frames-per-second camera. The time taken for 25 full oscillations, $N$, will then be counted using the video, starting from the first minimum. This value will then be divided by 25 to obtain the average period of one oscillation, reducing any random errors.

**Controlled Variables:**

1. The mass (kg) used on the free-end
   - Different masses may lead to different periods of oscillation. This is also backed by the presence of mass in equation 2.12. Owing to the pandemic, laboratory weights were unavailable. Hence, the same 0.250kg cylindrical padlock is used throughout the experiment as a mass.
2. The position and orientation of the mass on the cantilever
   - The padlock's center of mass is determined using the Plumb Line method. It is secured on the end of the cantilever such that the center of mass lies directly above the free end. It is ensured that the shackle of the lock always faces towards the table, along the cantilever.
3. The material, width (m), and height(m) of the cantilever
   - The same steel 0.600m cantilever is used throughout the experiment.
4. The camera used to record the video.
   - Different cameras may record videos with different settings, including different frame rates. This can affect the readings taken of $N$. The same 30 frames-per-second smartphone camera is used to record all the videos.
5. Wind
   - Wind from the ceiling fan may affect the period $T$ of the cantilever, hence during the experiment, fans were turned off and doors and windows were closed.

# 4 Apparatus

1. Kitchen scale
2. A mass (A padlock was used in this case)
3. 60cm Steel Ruler
4. 3-inch G-Clamp
5. Table
6. 1080-by-1920 pixel resolution, 30 frames-per-second, smartphone camera
7. Smartphone Tripod Stand with adjustable height.
8. Measuring tape
9. Tape and rubber bands.

# 5 Methodology

1. Measure and record the mass $m$ (in kg) used using the kitchen scale. Measure and record the width $b$ and thickness $h$ of the ruler using the measuring tape.
2. Secure the mass to the end of the ruler using tape and rubber bands.
3. Secure the ruler to the table using the G-Clamp, such that the mass is on the free end. Ensure that the length that is freely suspended, $l$, is 0.600m. This can be done based on the marking on the ruler.
4. Ensure that the system is at rest (i.e., no oscillation or motion is taking place).
5. Use the measuring tape to measure the distance from the ground to the equilibrium position of the free end, $y_1$ .
6. Attach the smartphone to the smartphone tripod and adjust the height such that the camera is at $y_1$ to minimize parallax. Use the measuring tape to position the camera 1m away perpendicularly from the cantilever. Start recording.
7. Using the measuring tape, displace the free-end upwards by 0.100m and release. Stop recording after 60s.
8. Using the videos, record the time taken for 25 complete oscillations into the table.
9. Repeat steps 4 to 8 four further times.
10. Repeat steps 3 to 9 for different oscillating lengths $l$ = 0.300m, 0.350m, 0.400m, 0.450m, 0.500m, 0.550m, 0.600m

While unlikely to happen, a pillow was placed underneath the oscillating length in case the mass detaches or the ruler breaks.

Ethical and environmental concerns are not relevant to this investigation.

# 6 Diagram

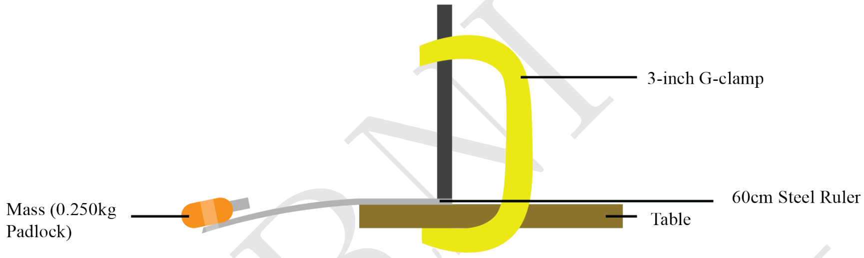



Figure 2. Side View of Experimental Set-Up. Image created with Adobe Illustrator.

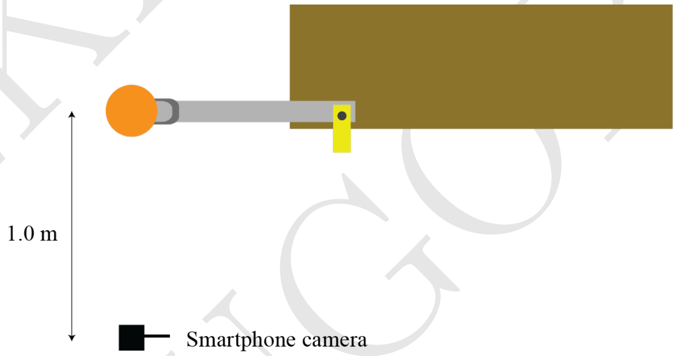



Figure 3. Top View of Experimental Set-Up. Image created with Adobe Illustrator.

# 7 Raw Data

Mass of padlock used, $m$: $(0.250 \pm 0.025)\,\text{kg}$.

Width of cantilever, $b$: $(0.0300 \pm 0.0005)\,\text{m}$.

Thickness of cantilever, $h$: $(0.001\,500 \pm 0.000\,005)\,\text{m}$.

Table 1. Measured time taken for 25 oscillations, $N$, for each corresponding oscillating length $l$.

| | $N(\text{s})\pm0.017$ | | | | |
|---|---|---|---|---|---|
| $l(\text{m})\pm0.0005$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $N_5$ |
| 0.300 | 6.976 | 6.539 | 6.842 | 6.812 | 7.323 |

| | | | | | |
|---|---|---|---|---|---|
| 0.350 | 9.394 | 9.195 | 9.840 | 8.547 | 8.620 |
| 0.400 | 10.518 | 10.513 | 10.843 | 10.996 | 11.253 |
| 0.450 | 12.521 | 12.552 | 12.271 | 12.531 | 12.528 |
| 0.500 | 14.534 | 14.691 | 14.718 | 14.715 | 14.689 |
| 0.550 | 16.955 | 16.935 | 16.936 | 16.924 | 16.917 |
| 0.600 | 19.146 | 19.050 | 19.101 | 19.014 | 18.988 |

The mass of the padlock was measured with a kitchen scale of smallest division 0.050 kg, hence the uncertainty was taken to be its half: $\frac{0.05\,\text{kg}}{2} = 0.025\,\text{kg}$. The width of the cantilever was measured with another ruler, hence the uncertainty was taken to be half of the smallest division of this ruler: $\frac{0.10\,\text{cm}}{2} = 0.0005\,\text{m}$. The thickness of the ruler was measured with a micrometer screw gauge, hence the uncertainty was taken to be half the smallest division: $\frac{0.01\,\text{mm}}{2} = 0.000\,005\,\text{m}$. The uncertainty of $N$ is taken to be half of the time between two frames of the 30 frames-per-second video: $\frac{1 \div 30\,\text{s}^{-1}}{2} \approx 0.017\,\text{s}$.

# 8 Processed Data

To process the above data, the values of the time taken for 25 oscillations for each length were first averaged. These were then divided by 25 to obtain the average period of a single oscillation, $T_{avg}$ for each length. The uncertainty $\Delta T_{avg}$ was obtained by first dividing the range of values for the time taken for 25 oscillations by 2 and then dividing this by 25 to obtain the uncertainty for the average period. Sample calculations are shown below.

Table 2. Average Period of Oscillation $T_{avg}$ for Corresponding Oscillation Length $l$.

| $l$(m)$\pm$0.0005 | $N_{avg}$(s) | $T_{avg}$(s) | $\Delta T_{avg}$(s) |
|---|---|---|---|
| 0.300 | 6.898 | 0.276 | 0.016 |
| 0.350 | 9.119 | 0.365 | 0.027 |
| 0.400 | 10.825 | 0.433 | 0.015 |
| 0.450 | 12.481 | 0.499 | 0.006 |
| 0.500 | 14.669 | 0.587 | 0.004 |
| 0.550 | 16.933 | 0.677 | 0.001 |

| | | | |
|---|---|---|---|
| 0.600 | 19.060 | 0.762 | 0.004 |

Sample Calculations:

$$N_{avg} = \frac{N_1 + N_2 + N_3 + N_4 + N_5}{5} \tag{8.1}$$

$$= \frac{6.976 + 6.539 + 6.842 + 6.812 + 7.323}{5} = 6.898\,\text{s} \tag{8.2}$$

$$T_{avg} = \frac{N_{avg}}{25} \tag{8.3}$$

$$= \frac{6.898}{25} = 0.276\,\text{s} \tag{8.4}$$

$$\Delta T_{avg} = \frac{N_{max} - N_{min}}{2 \times 25} \tag{8.5}$$

$$= \frac{(7.323 + 0.017) - (6.539 - 0.017)}{50} = 0.016\,\text{s} \tag{8.6}$$

Based on the theoretical model presented in Section 2, the graph of $T^2$ against $l^3$ should be linear. In Table 3 below, the cubes of the oscillation length and square of the average period as well as their respective uncertainties are computed based on the values in Table 2.

Table 3. Values of Average Period Squared ${T_{avg}}^2$ for corresponding Oscillation Length Cubed $l^3$.

| $l^3$(m$^3$) | $\Delta l^3$ (m$^3$) | $T_{avg}^2$(s$^2$) | $\Delta T_{avg}^2$(s$^2$) |
|---|---|---|---|
| 0.0270 | 0.0001 | 0.076 | 0.009 |
| 0.0429 | 0.0002 | 0.133 | 0.019 |
| 0.0640 | 0.0002 | 0.187 | 0.013 |
| 0.0911 | 0.0003 | 0.249 | 0.006 |
| 0.1250 | 0.0004 | 0.344 | 0.005 |
| 0.1664 | 0.0005 | 0.459 | 0.002 |
| 0.2160 | 0.0005 | 0.581 | 0.006 |

Sample Calculations:

$$l^3 = (0.300\,\mathrm{m})^3 = 0.027\,\mathrm{m}^3 \tag{8.7}$$

$$\Delta l^3 = \frac{\Delta l}{l} \times 3 \times l^3 \tag{8.8}$$

$$= \frac{0.0005\,\mathrm{m}}{0.300\,\mathrm{m}} \times 3 \times 0.0270\,\mathrm{m}^3 \approx 0.0001\,\mathrm{m}^3 \tag{8.9}$$

$$T_{avg}^2 = (0.276\,\mathrm{s})^2 \approx 0.076\,\mathrm{s}^2 \tag{8.10}$$

$$\Delta T_{avg}^2 = \frac{\Delta T_{avg}}{T_{avg}} \times 2 \times T_{avg}^2 \tag{8.11}$$

$$= \frac{0.016\,\mathrm{s}}{0.276\,\mathrm{s}} \times 2 \times 0.076\,\mathrm{s}^2 \approx 0.009\,\mathrm{s}^2 \tag{8.12}$$

Note that horizontal error bars are not visible in Figure 4 as the uncertainties for $l^3$ is negligibly small.

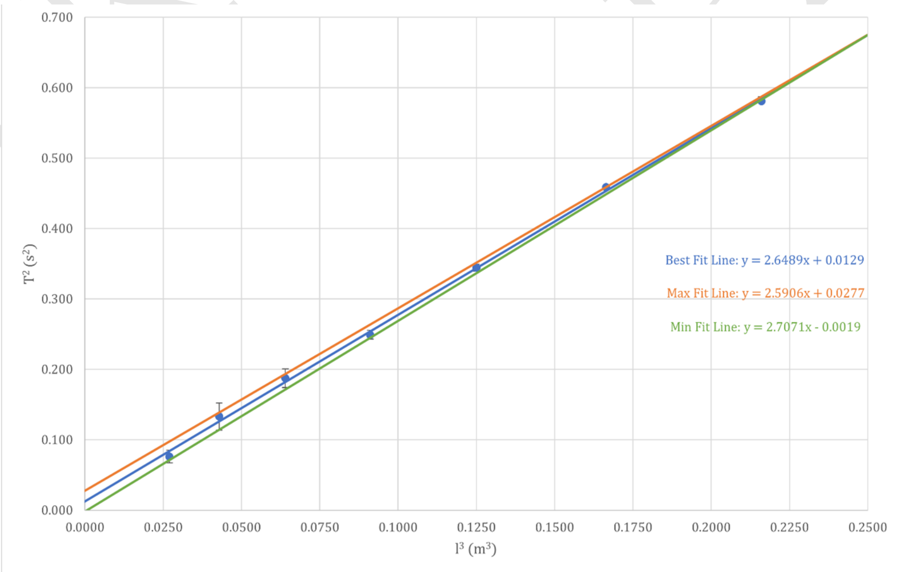



Figure 4. Period of Oscillation Squared ($T^2$) Against Oscillating Length Cubed ($l^3$)

# 9 Analysis

It was hypothesized earlier that $T$ will correlate positively to $l$. It is clear even from the data in table 2 that there is such a positive trend: as $l$ increases, $T$ increases. Furthermore, based on graph 1, $T^2$ is linearly and positively proportional to $l^3$. However, the best fit line is shifted slightly from the origin $(0, 0)$, as can be seen from the non-zero y-intercepts. Since the minimum fit line and maximum fit line have y-intercepts below and above the origin respectively, it is reasonable to say that $T^2$ is directly proportional to $l^3$. This conclusion agrees with the hypothesis that the square of the period, $T^2$, will be directly proportional to the cube of the oscillating length, $l^3$.

It should be noticed that the precision of $T^2$ tended to be lower for shorter values of $l^3$ compared to longer values. This is noticeable from the lower $\Delta T^2_{avg}$values for higher values of $l^3$ in table 3 and the lack of visible error bars for $T^2$ for higher values of $l^3$ in graph 1.

This discrepancy could be due to the uncertainty in measurement disproportionally affecting the faster periods of shorter oscillating lengths. Despite going through the video frame by frame, the higher speed of the free-end for shorter oscillating length meant it was difficult to obtain the exact value of time for the starting and stopping for the 25 oscillations. This moment would often happen between frames.

The gradients of the maximum fit line and the minimum fit lines are given by Excel to be

$$P_{max} = 2.591\,\mathrm{s^2\,m^{-3}} \tag{9.1}$$
$$P_{min} = 2.707\,\mathrm{s^2\,m^{-3}} \tag{9.2}$$

Hence:

$$P_{best} = (2.649 \pm 0.058)\,\mathrm{s^2\,m^{-3}} \tag{9.3}$$

According to the theoretical model derived in section 2,

$$P = \frac{16\pi^2 m}{Ebh^3} = (2.649 \pm 0.058)\,\mathrm{s^2\,m^{-3}} \tag{9.4}$$

Substituting the measured values of $m$, $b$ and $h$:

$$\frac{16\pi^2 \times (0.250 \pm 0.025)\,\mathrm{kg}}{E \times (0.0300 \pm 0.0005)\,\mathrm{m} \times ((0.001\,500 \pm 0.000\,005)\,\mathrm{m})^3} = (2.649 \pm 0.058)\,\mathrm{s^2\,m^{-3}} \tag{9.5}$$

The value of $E$ is then:

$$E = \frac{16\pi^2 \times (0.250 \pm 0.025)\,\mathrm{kg}}{(0.0300 \pm 0.0005)\,\mathrm{m} \times ((0.001\,500 \pm 0.000\,005)\,\mathrm{m})^3 \times (2.649 \pm 0.058)\,\mathrm{s^2\,m^{-3}}} \tag{9.6}$$
$$E = \frac{16\pi^2 \times (0.250\,\mathrm{kg} \pm 10\%)}{(0.0300\,\mathrm{m} \pm 2\%) \times (0.001\,500\,\mathrm{m} \pm 0.3\%)^3 \times (2.649\,\mathrm{s^2\,m^{-3}} \pm 2.2\%)} \tag{9.7}$$
$$E = 147 \times 10^9\,\mathrm{kg\,m^{-1}\,s^{-2}} \pm 15.1\% = 147 \times 10^9\,\mathrm{N\,m^{-2}} \pm 15.1\% \tag{9.8}$$

The uncertainty of $E$ here is rather large due to the large relative measurement uncertainty of $m$. Nonetheless the estimated value is within 18.3% of the literature value, $180 \times 10^9\,\mathrm{N\,m^{-2}}$ (ToolBox). With more appropriate lab apparatus such as an electronic mass balance to measure mass $m$, this can be a fairly accurate method of obtaining the

Young's Modulus of materials. This could be especially useful in situations where conventional methods like the static deflection model of determining the Young's Modulus are not useful due to limitations such as cantilever size (Whitney).

Initially, due to pandemic-related constraints, the value of $h$ was measured with the same ruler that was used to measure $b$. This resulted in a large relative measurement uncertainty of 33% for $h$. Furthermore, the impact of this uncertainty was further exacerbated as $h$ was raised to the power of 3. The final value had a calulated uncertainty greater than 100% and that would have not been a suitable quantity to compare against literature values.

## 10 Evaluation

| **Shortcomings** | **Suggested Improvements** |
|---|---|
| As previously mentioned, a camera was used to allow for more precise measurements of the period of oscillation with minimal human error due to reaction time. However, the framerate used was relatively low. This resulted in it becoming difficult to obtain precise measurements as oscillations often reached their end in between frames. | A camera with a higher framerate can be used allowing for more precise measurements to be taken for the period of oscillation |
| The free end of the cantilever was initially displaced by hand. This might have led to human error. | The initial amplitude should not theoretically directly affect the period of oscillation. This is evidenced by $\omega$ not depending on the initially amplitude A. However, this theoretical expression need not be completely accurate and hence the initial displacement should be controlled as well as possible. This could be improved by using a motor that can hold the pendulum at a fixed displacement before retracting and releasing it to oscillate. |
| Air Resistance was ignored. Longer cantilever lengths might have larger air resistance acting on it, affecting the period disproportionately | Ideally this experiment should be conducted in a vacuum chamber. However, given the low surface area of the cantilever, air resistance is unlikely to have substantially affected the results. |

| | |
|---|---|
| The range of lengths of cantilever used was quite small. While this investigation suggests that the theoretical model derived at the beginning is accurate, it is possible that it is not as accurate for longer oscillating lengths. The masses of longer suspended lengths will have greater effects on the motion. | Use a longer cantilever allowing for a greater range of oscillating lengths. This would allow a better comparison of the theoretical model to the experiment. |
| Damping was ignored. | While damping of the oscillation was ignored in the measuring of the period, it should not affect the period of the pendulum significantly. This is again due to the $\omega$ term not depending on the amplitude and hence no improvement is necessary. |

## 11 Conclusion and Future Work

In conclusion, based on the data collected and processed, the period of an oscillating cantilever is positively correlated to the oscillating length, such that $T^2 \propto l^3$. The theoretical model derived at the beginning of the investigation seems to largely fit the data, evidenced by the Young's Modulus obtained in the Analysis section.

In future experiments, a longer cantilever could be used, allowing for a larger range of data. A video camera with a higher framerate would also allow for more precise measurements of the period of oscillation, especially for shorter oscillation lengths.

In terms of the theoretical model, simple harmonic motion is a very rudimentary way of approaching the cantilever problem. In a realistic situation, like with the Marina Bay Sands, it may be more appropriate to bring in the beam equation to model the movement more accurately by including the effect of mass of the cantilever itself. The beam equation suggests that cantilever beams oscillate in "modes". Unless the experiment is set up appropriately, it is possible for several such modes to be excited at once (Whitney), which can affect the period. The effects of damping, if any, can also be considered, in terms of both the simple harmonic motion model and the beam equation model.

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